Lecture notes on Kinematics

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1 INTRODUCTION

Design and analysis are two vital tasks in engineering. 

Design process means the synthesis during the proposal phase the size, shape, material properties and arrangements of the parts are prescribed in order to fulfil the required task.

Analysis is a technique or rather set of tools allowing critical evaluation of existing or proposed design in order to judge its suitability for the task.

Thus synthesis is a goal that can be reached via analysis.

Mechanical engineer deals with many different tasks that are in conjunction to diverse working processes referred to as a technological process.

Technological processes involve transportation of material, generation and transformation of energy, transportation of information. All these processes require mechanical motion, which is carried out by machines.

To be able to create appropriate design of machine and mechanism the investigation of relation between the geometry and motion of the parts of a machine/mechanism and the forces that cause the motion has to be carried out. Thus the mechanics as a science is involved in the design process. Mechanics represents the science that includes Statics, Dynamics, and Mechanics of Materials.

Statics provides analysis of stationary systems while Dynamics deals with systems that change with time and as Euler suggested the investigation of motion of a rigid body may be separated into two parts, the geometrical part and the mechanical part. Within the geometrical part Kinematics the transference of the body from one position to the other is investigated without respect to the causes of the motion. The change is represented by analytical formulae.

Thus Kinematics is a study of motion apart from the forces producing the motion that is described by position, displacement, rotation, speed, velocity, and acceleration.

In Kinematics we assume that all bodies under the investigation are rigid bodies thus their deformation is negligible, and does not play important role, and the only change that is considered in this case is the change in the position.

Terminology that we use has a precise meaning as all the words we use to express ourselves while communicating with each other. To make sure that we do understand the meaning we have a thesaurus/glossary available. It is useful to clarify certain terms especially in areas where the terminology is not very clear.

Structure represents the combination of rigid bodies connected together by joints with intention to be rigid. Therefore the structure does not do work or transforms the motion. Structure can be moved from place to place but it does not have an internal mobility (no relative motion between its members).

Machines & Mechanisms – their purpose is to utilize relative internal motion in transmitting power or transforming motion.

Machine – device used to alter, transmit, and direct forces to accomplish a specific objective.

Mechanism – the mechanical portion of a machine that has the function of transferring motion and forces from power source to an output. Mechanism transmits motion from drive or input link to the follower or the output link.
Planar mechanism – each particle of the mechanism draws plane curves in space and all curves lie in parallel planes. The motion is limited to two-dimensional space and behaviour of all particles can be observed in true size and shape from a single direction. Therefore all motions can be interpreted graphically. Most mechanisms today are planar mechanism so we focus on them.

Spherical mechanism – each link has a stationary point as the linkage moves and the stationary points of all links lie at a common location. Thus each point draws a curve on the spherical surface and all spherical surfaces are concentric.

Spatial mechanism – has no restriction on the relative motion of the particles. Each mechanism containing kinematical screw pair is a spatial mechanism because the relative motion of the screw pair is helical.

The mechanism usually consists of:

Frame – typically a part that exhibits no motion

Links – the individual parts of the mechanism creating the rigid connection between two or more elements of different kinematic pair. (Springs cannot be considered as links since they are elastic.)

Kinematic pair (KP) represents the joint between links that controls the relative motion by means of mating surface thus some motions are restricted while others are allowed. The number of allowed motions is described via mobility of the KP. The mating surfaces are assumed to have a perfect geometry and between mating surfaces there is no clearance.

Joint – movable connection between links called as well kinematic pair (pin, sliding joint, cam joint) that imposes constrains on the motion

Kinematic chain is formed from several links movably connected together by joints. The kinematic chain can be closed or opened according to organization of the connected links.

Simple link – a rigid body that contains only two joints

Complex link – a rigid body that contains more than two joints

Actuator – is the component that drives the mechanism

Last year we started to talk about the foundation of Mechanics – Statics and later on about the transfer of the forces and their effect on the elements of the structure/machine. Our computation of the forces was based on the Statics only and at the beginning we assumed that the forces exist on the structure or are applied very slowly so they do not cause any dynamical effect on the structure. This situation is far from real world since there is nothing stationary in the world. (Give me a fixed point and I’ll turn the world. Archimedes 287 BC – 212 BC Greek mathematician, physicist )

Kinematics deals with the way things move. It is a study of the geometry of motion that involves determination of position, displacement, speed, velocity, and acceleration.

This investigation is done without consideration of force system acting on an actuator. Actuator is a mechanical device for moving or controlling a mechanism or system. Therefore the basic quantities in Kinematics are space and time as defined in Statics.
Kinematics describes the motion of an object in the space considering the time dependency. The motion is described by three kinematics quantities:

The position vector gives **the position** of a particular point in the space at the instant.
The time rate of change of the position vector describes **the velocity** of the point.

**Acceleration** – the time rate of change of the velocity

All quantities – position, velocity, and acceleration are vectors that can be characterized with respect to:

*Change of a scalar magnitude* – uniform motion
  - Uniformly accelerated motion
  - Non-uniformly accelerated motion
  - Harmonic motion

**Character of the trajectory** - 3D (motion in the space)
  - 2D (planar motion)

**The type of trajectory** can be specified as:
  - Rectilinear motion
  - Rotation
  - Universal planar motion
  - Spherical motion
  - Universal space motion
  - Complex motion

The set of independent coordinates in the space describes the position of a body as a time-function thus defines the **motion of a body**.

The **number of independent coordinates** corresponds to the degree of freedom of the object or set of coupled bodies and it is expressed as the mobility of the object.

**Mobility** – the number of degrees of freedom possessed by the mechanism. The number of independent coordinates (inputs) is required to precisely position all links of the mechanism with respect to the reference frame/coordinate system.

For planar mechanism:
\[ i = 3(n - 1) - \sum j_{DOF} \]

For space mechanism:
\[ i = 6(n - 1) - \sum j_{DOF} \]

**Kinematical diagram** – is “stripped down” sketch of the mechanism (skeleton form where only the dimensions that influence the motion of the mechanism are shown).

**Particle** – is a model body with very small/negligible physical dimensions compared to the radius of its path curvature. The particle can have a mass associated with that does not play role in kinematical analysis.
**How to find the degree of freedom?**

1. Consider an unconstrained line moving in the space

   ![Diagram of a line](image1)

   The non-penetrating condition between points A, B 
   \[ \overline{AB} = \text{const.} = l \]

   number of degrees of freedom for a line in 3D:
   two points \( \implies 2 \times 3 = 6 \text{ DOF} \)
   non-penetrating condition: \( \overline{AB} = l \)
   Thus \( i = 6 - 1 = 5 \text{ DOF} \)
   Conclusion: A free link AB has five degrees of freedom when moving in the space.

2. Consider an unconstrained body in the space

   ![Diagram of a body](image2)

   How many points will describe position of a body?

   Three points: \( 3 \times 3 = 9 \text{ DOF} \)
   Non-penetrating condition (assume rigid body):
   \[ \overline{AB} = \text{const.}; \overline{AC} = \text{const.}; \overline{BC} = \text{const.} \]
   thus \( m = 3 \)
   and \( i = 3 \times 3 - 3 = 6 \text{ DOF} \)

   Conclusion: A free solid body has six degrees of freedom when moving in the space.

To be able to evaluate DOF the **kinematical diagram** of the mechanism has to be created.

**Diagrams** should be drawn to scale proportional to the actual mechanism in the given position.

The convention is to number links starting with the reference frame as number one while the joints should be lettered.

The adopted strategy should consist of identifying on the real set of bodies:
- the frame, the actuator, and all the other links
- all joints
- any points of interest
and draw the kinematical diagram according to the convention.

Once we evaluated the mobility (degrees of freedom) we can identify the corresponding set of independent coordinates (parameters) and start the kinematical analysis of the mechanism.
proceeding through the sub-task:

a) define the reference frame (basic space in which the motion will be described)
b) define the position of a point/particle with respect to the reference frame
c) describe the type of motion (constrained or unconstrained)
d) write the non-penetrating conditions
e) define the independent coordinates
f) find the velocity and acceleration

\[ i = 3(6 - 1) - (5 \cdot 2 + 3 \cdot 1) \]

The kinematical analysis of the whole set of connected bodies can be done if we would be able
to describe the motion of each segment/body and then identify the kinematical quantities at the
point of interest in the required position or time.
Thus let’s start with the Kinematics of a Particle that is shown on the diagram as a point.
2 KINEMATICS OF A PARTICLE

The position vector \( \mathbf{r}_A \) describes the position of a particle/point \( A \), with respect to the reference frame (CS \( x, y, z \)).

Character of a position vector depends on the arbitrary coordinate system.

At the instant the point \( A \) has a position

\[
\mathbf{r}_A = \mathbf{r}(t) = f(t)
\]

during the time interval \( \Delta t \) the point moves to a new position \( A_1 \) that can be described by a position vector \( \mathbf{r}_{A1} \)

\[
\mathbf{r}_{A1} = \mathbf{r}_A + \Delta \mathbf{r}
\]

where: \( \Delta \mathbf{r} \) represents the position vector increment in time interval

\( \Delta s \) represents the trajectory increment in time interval.

The Distance represents the measure of the point instant position with respect to the origin.

Trajectory/path of the particular point is the loci of all instant positions of that point.

The unit vector of the trajectory:

\( \mathbf{\tau} \) unit vector in the tangent direction

\[
\mathbf{\tau} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{r}}{\Delta s} = \frac{d\mathbf{r}}{ds}
\]

then

\[
\mathbf{\tau} = \frac{d}{ds} (xi + yj + zk) = \frac{dx}{ds}i + \frac{dy}{ds}j + \frac{dz}{ds}k
\]

where

\[
\frac{dx}{ds} = \cos \alpha; \quad \frac{dy}{ds} = \cos \beta; \quad \frac{dz}{ds} = \cos \gamma;
\]

are the directional cosines of the tangent to the trajectory, and angles \( \alpha, \beta, \gamma \) are the angles between axes \( x, y, z \) and the tangent vector \( \mathbf{\tau} \).

\( \mathbf{n} \) unit vector in the normal direction to the trajectory has positive orientation towards the centre of the trajectory curvature

\[
\mathbf{n} = \lim_{\Delta \theta \to 0} \frac{\Delta \mathbf{\tau}}{\Delta \theta} = \frac{d\mathbf{\tau}}{d\theta}
\]

taking into account the trajectory curvature radius \( R \) then

\[
ds = R.d\theta \quad \text{and} \quad d\theta = \frac{ds}{R}
\]

thus \( \mathbf{n} = \frac{d\mathbf{\tau}}{ds} = R.\frac{d\mathbf{\tau}}{ds} \). Substituting for \( \mathbf{\tau} \) we get

\[
\mathbf{n} = R \frac{d^2 \mathbf{r}}{ds^2} = R \frac{d^2 x}{ds^2}i + R \frac{d^2 y}{ds^2}j + R \frac{d^2 z}{ds^2}k
\]
where: \[ R \frac{d^2x}{ds^2} = \cos \alpha_n ; \quad R \frac{d^2y}{ds^2} = \cos \beta_n ; \quad R \frac{d^2z}{ds^2} = \cos \gamma_n \] are the directional cosines of the normal to the trajectory, and 
\[ \alpha_n, \beta_n, \gamma_n \] are the angles between axes \( x, y, z \) and the normal.

In case of 3D motion the trajectory is a 3D curve thus third unit vector in bi-normal direction has to be defined:

\[ b \ldots \text{unit vector in the bi-normal direction to the trajectory is oriented in a way that} \]

the positive direction of bi-normal vector forms together with normal and tangent right oriented perpendicular system.

\[ b = \tau \times n \]

### 2.1 VELOCITY

Is the time rate of change of the positional vector.

The average velocity of change is defined as \[ v_{avr} = \frac{\Delta r}{\Delta t} \]

Our interest is to find an instant velocity, that represents the limit case of average velocity. The time limit for computation of the instant velocity is approaching zero.

The instant velocity is define as: \[ v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt} = \dot{r} \]

**What is the direction of the instant velocity?**

A common sense or rather to say intuition suggests that the velocity has the tangent direction to the trajectory. So let’s prove this statement mathematically:

\[ v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} \cdot \frac{\Delta s}{\Delta s} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta s} \cdot \frac{\Delta s}{\Delta t} = \frac{dr}{ds} \cdot \frac{ds}{dt} = \tau \cdot \dot{s} = \tau \cdot v \]

since \[ \dot{s} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = v \]

Having a positional vector defined as: \[ r = xi + yj + zk \]

then the velocity can be described by its components, since:

\[ v = \frac{dr}{dt} = \frac{d}{dt} (xi + yj + zk) = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k = v_xi + v_yj + v_zk \]

where \( v_x, v_y, v_z \) are components of the velocity in the direction of the axes of coordinate system.

The magnitude/modulus of velocity: \[ |v| = \sqrt{v_x^2 + v_y^2 + v_z^2} \]

with directional cosines: \[ \cos \alpha_v = \frac{v_x}{|v|} ; \quad \cos \beta_v = \frac{v_y}{|v|} ; \quad \cos \gamma_v = \frac{v_z}{|v|} \]
2.2 ACCELERATION

The acceleration of a change of position is the time rate of change of velocity. To derive the expression for acceleration we need to draw velocity vector diagram so called velocity hodograph.

Constructing hodograph based on the knowledge of path of a point and its velocity in particular position A and A₁:

Let's have arbitrary point P through which both velocities \( v_A \) and \( v_{A1} \) will pass. The end points of their vectors are creating the desired curve hodograph.

Based on hodograph \( v_{A1} = v_A + Δv \)

The average acceleration is given as \( a_{avr} = \frac{Δv}{Δt} \)

The instant acceleration is given as the limit value of average acceleration for time interval \( Δt \rightarrow 0 \)

\[
a = \lim_{Δt \to 0} \frac{Δv}{Δt} = \frac{dv}{dt} = \ddot{v} = \dddot{r}
\]

The direction of acceleration can be found from

\[
a = \frac{dv}{dt} = \frac{d}{dt} (\tau \cdot v) = \frac{d\tau}{dt} v + \tau \cdot \frac{dv}{dt}
\]

Thus

\[
a = \frac{d\tau}{dt} \cdot \frac{ds}{dt} \cdot v + \frac{dv}{dt} = \frac{d\tau}{dt} \cdot \frac{ds}{dt} \cdot v + \tau \cdot \frac{dv}{dt} = \frac{d\tau}{dt} v^2 + \tau \cdot \frac{dv}{dt}
\]

Since the direction of the normal is given as \( n = R \frac{d\tau}{ds} \) then we can substitute \( \frac{d\tau}{ds} = \frac{n}{\rho} \) where \( \rho \) represents the radius of the curvature at the instant. Therefore:

\[
a = n \cdot \frac{v^2}{\rho} + \tau \cdot \frac{dv}{dt} = n \cdot \frac{1}{\rho} \cdot \dot{s}^2 + \tau \cdot \ddot{s} = a_n + a_t
\]

Where \( a_n = n \cdot \frac{v^2}{\rho} \) is the acceleration in normal direction, and

\( a_t = \tau \cdot \ddot{s} \) is the tangential component of the acceleration.
The direction of normal acceleration is always oriented to the center of instant curvature of the trajectory. The tangent component of acceleration captures the change of magnitude of a velocity while the normal component captures the change of direction of a velocity.

The resultant acceleration forms an angle $\beta$ with normal direction: $\tan \beta = \frac{|a|}{|a_n|}$

Thus the acceleration expressed in the rectangular coordinate system would have form:

$$a = \frac{dv}{dt} = \frac{d}{dt}(v_x i + v_y j + v_z k) = a_x i + a_y j + a_z k$$

$$a = \frac{d}{dt}(\dot{x} i + \dot{y} j + \dot{z} k) = \ddot{x} i + \ddot{y} j + \ddot{z} k$$

and the magnitude of acceleration: $|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Orientation of the final acceleration is given by directional cosines:

$$\cos \alpha_a = \frac{a_x}{|a|}; \cos \beta_a = \frac{a_y}{|a|}; \cos \gamma_a = \frac{a_z}{|a|}$$

and at the same time $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

Precise description of the motion of a particle is given by function capturing all kinematic quantities $f(\mathbf{r}, \mathbf{v}, \mathbf{a}, a_n, t) = 0$

### 2.2.1 Classification of motion

Consider the motion of the particle along the straight line (in the direction of x-axis). The tangential component of acceleration captures the change of velocity magnitude, thus it can be used to distinguish motion as:

**Uniform motion**

Mathematical description: $a_t = 0$ thus $a_t = \frac{dv}{dt} = 0$ that implies $v = \text{const.}$

In case that the tangent takes the direction of axis $x$ then $v_x = \text{const.}$ and equation $v_x = \frac{dx}{dt}$ represents the simple differential equation solved by separation of variables $v_x \int_0^t dt = \int_{x_0}^x dx$, thus giving the solution $x = x_0 + v_x \cdot t$
**Uniformly accelerated/decelerated motion**

Mathematical description: \( a_t = \text{const.} \)

In case that the tangent takes the direction of axis \( x \) then \( a_x = \text{const.} \) and \( a_x = \frac{dv_x}{dt} \) thus

\[ a_x \int_0^t dt = \int_{v_0}^{v} dv_x \]

leading to solution

\[ v_x = v_0 + a_x t \]

at the same time 

\[ v_x = \frac{dx}{dt} \]

therefore

\[ \int_0^t (v_0 + a_x t) dt = \int_{v_0}^{v} dv_x \]

that gives the solution:

\[ x = x_0 + v_0 t + \frac{1}{2} a_x t^2 \]

The solution lead to an equation of trajectory of the point expressed as a function of time.

**Non-uniformly accelerated motion**

Mathematical description: \( a_t = a_x = a_0 \pm kt \) (the function could be define differently)

Thus \( a_x \neq \text{const.} \) \( \frac{dv_x}{dt} \) therefore

\[ \int_0^t (a_0 \pm kt) dt = \int_{v_0}^{v} dv_x \]

with solution

\[ v_x = v_0 + a_0 t \pm \frac{1}{2} kt^2 \]

since \( v_x = \frac{dx}{dt} \) then

\[ \int_0^t \left( v_0 + a_0 t \pm \frac{1}{2} kt^2 \right) dt = \int_{x_0}^{x} dx \]

that gives the trajectory equation

\[ x = x_0 + v_0 t + \frac{1}{2} a_0 t^2 \pm \frac{1}{6} kt^3 \]
**Motion with other changes of kinematic quantities**

In this case the acceleration is given as a function of other quantities \( a = f(r, v, t) \)

### 2.3 ORTHOGONAL TRANSFORMATION

You can see very clearly that velocity and acceleration directly depend on the positional vector. The positional vector form will vary according to the type of the coordinate system. Thus in rectangular coordinate system \((x, y, z)\) the kinematical quantities in vector form are:

\[
\mathbf{r} = r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k} \quad \text{(positional vector)}
\]

\[
\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} = \frac{dt}{dt} \left( r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k} \right) \quad \text{(velocity)}
\]

\[
\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} = \frac{d^2}{dt^2} \left( r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k} \right) \quad \text{(acceleration)}
\]

Let the point \( A \) be attached to the moving coordinate system \( x_2, y_2, z_2 \) with its origin coinciding with fixed coordinate system \( x_1, y_1, z_1; \) \( O_1 \equiv O_2 \)

The vector form of a position of the point \( A \) in \( CS1 \):

\[
\mathbf{r}_1^A = r_{1x}^A \mathbf{i} + r_{1y}^A \mathbf{j} + r_{1z}^A \mathbf{k}
\]

and in \( CS2 \):

\[
\mathbf{r}_2^A = r_{2x}^A \mathbf{i} + r_{2y}^A \mathbf{j} + r_{2z}^A \mathbf{k}
\]

in matrix form:

\[
\begin{bmatrix} x_1^A \\ y_1^A \\ z_1^A \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2^A \\ y_2^A \\ z_2^A \end{bmatrix}
\]

or

\[
\mathbf{r}_1^{AT} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and} \quad \mathbf{r}_2^{AT} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}
\]

To express the positional vector \( r_2^A \) in \( CS1 \) the vector has to be transformed. This process is called **Orthogonal Transformation of Vector Quantities**

#### 2.3.1 Orthogonal Transformation of Vector Quantities

The mathematical operation is using matrix form of a vector.

The \( CS1 \) is associated with basic frame/space that serves as a fixed reference frame that does not move.

The moving point \( A \) is connected to the \( CS2 \), which moves with respect to the reference frame.

Thus position of the point \( A \) in \( CS1 \) is

\[
\mathbf{r}_1^A = x_1^A \mathbf{i} + y_1^A \mathbf{j} + z_1^A \mathbf{k} \quad \text{and in} \ CS2
\]

\[
\mathbf{r}_2^A = x_2^A \mathbf{i} + y_2^A \mathbf{j} + z_2^A \mathbf{k}
\]
How do we interpret positional vector $\mathbf{r}_2^A$ in CS1? The task is to project vector $\mathbf{r}_2^A$ into CS1.

Thus projecting vector $\mathbf{r}_2^A$ into the $x_1$ direction:

$$x_1^A = \mathbf{r}_2^A \cdot i_1 = (x_1^A i_2 + y_1^A j_2 + z_1^A k_2) \cdot i_1$$

where:

$$\cos \alpha_1 = \frac{i_1}{i_2} \Rightarrow i_1 = i_2 \cdot \cos \alpha_1$$

$$\cos \alpha_2 = \frac{i_2}{j_2} \Rightarrow i_1 = j_2 \cdot \cos \alpha_2$$

$$\cos \alpha_3 = \frac{i_1}{k_2} \Rightarrow i_1 = k_2 \cdot \cos \alpha_3$$

Thus

$$x_1^A = x_2^A i_2 \cdot i_1 + y_2^A j_2 \cdot i_1 + z_2^A k_2 \cdot i_1 = x_2^A i_2 i_1 + y_2^A j_2 j_1 + z_2^A k_2 k_1 \cdot i_1 = x_2^A \cos \alpha_1 + y_2^A \cos \alpha_2 + z_2^A \cos \alpha_3$$

$$y_1^A = \mathbf{r}_2^A \cdot j_1 = (x_2^A i_2 + y_2^A j_2 + z_2^A k_2) \cdot j_1 = x_2^A \cos \beta_1 + y_2^A \cos \beta_2 + z_2^A \cos \beta_3$$

$$z_1^A = \mathbf{r}_2^A \cdot k_1 = (x_2^A i_2 + y_2^A j_2 + z_2^A k_2) \cdot k_1 = x_2^A \cos \gamma_1 + y_2^A \cos \gamma_2 + z_2^A \cos \gamma_3$$

Rewriting these three equations in matrix form will give

$$\begin{bmatrix}
x_1^A \\
y_1^A \\
z_1^A
\end{bmatrix} =
\begin{bmatrix}
\cos \alpha_1 & \cos \alpha_2 & \cos \alpha_3 \\
\cos \beta_1 & \cos \beta_2 & \cos \beta_3 \\
\cos \gamma_1 & \cos \gamma_2 & \cos \gamma_3
\end{bmatrix}
\begin{bmatrix}
x_2^A \\
y_2^A \\
z_2^A
\end{bmatrix} \Rightarrow \mathbf{r}_1^A = C_{21} \cdot \mathbf{r}_2^A$$

where

$$C_{21} =
\begin{bmatrix}
\cos \alpha_1 & \cos \alpha_2 & \cos \alpha_3 \\
\cos \beta_1 & \cos \beta_2 & \cos \beta_3 \\
\cos \gamma_1 & \cos \gamma_2 & \cos \gamma_3
\end{bmatrix}$$

Analogically transformation from CS1 into CS2 gives: $\mathbf{r}_2^A = C_{12} \cdot \mathbf{r}_1^A$

where

$$C_{12} = C_{21}^T =
\begin{bmatrix}
\cos \alpha_1 & \cos \alpha_2 & \cos \alpha_3 \\
\cos \beta_1 & \cos \beta_2 & \cos \beta_3 \\
\cos \gamma_1 & \cos \gamma_2 & \cos \gamma_3
\end{bmatrix}^T$$

and $C_{21} \cdot C_{21}^T = I$

The planar motion is special case when at any time of the motion $z_1 = z_2$

$$\alpha_1 = \varphi; \quad \alpha_2 = \frac{\pi}{2} + \varphi; \quad \alpha_3 = \frac{\pi}{2}$$

$$\beta_1 = \frac{3\pi}{2} + \varphi; \quad \beta_2 = \varphi; \quad \beta_3 = \frac{\pi}{2}$$
\gamma_1 = \frac{\pi}{2}; \quad \gamma_2 = \frac{\pi}{2}; \quad \gamma_3 = 0

\text{and} \quad C_{21} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}

Once the positional vector is expressed in matrix form and the orthogonal transformation is used then the velocity and acceleration can be expressed in the same form.

### 2.3.2 Velocity in matrix form using the orthogonal transformation

Velocity is the first derivative of the positional vector

\[ v^A_i = \dot{r}^A_i = \frac{d}{dt} (C_{21} r^A_2) = C_{21} \dot{r}^A_2 + C_{21} \ddot{r}^A_2 \]

If the point A does not change its position with respect to the origin CS2 then \( r^A_2 = \text{const.} \) and therefore \( \dot{r}^A_2 = 0 \) and \( v^A_i = \dot{C}_{21} \cdot r^A_2 \)

### 2.3.3 Acceleration in matrix form using the orthogonal transformation

Acceleration is the first derivative of the velocity and a second derivative of the positional vector, thus

\[ a^A_i = \ddot{v}^A_i = \ddot{r}^A_i = \ddot{C}_{21} \cdot r^A_2 + \dot{C}_{21} \cdot \dot{r}^A_2 + \dot{C}_{21} \cdot \ddot{r}^A_2 + C_{21} \cdot \dddot{r}^A_2 \]

If the point A does not change its position with respect to origin CS2 then \( r^A_2 = \text{const.} \) and therefore \( \dot{r}^A_2 = 0 \) and \( \ddot{r}^A_2 = 0 \) thus giving \( a^A_i = \dddot{C}_{21} \cdot r^A_2 \)
2.4 PARTICLE IN CYLINDRICAL COORDINATE SYSTEM - $r, \varphi, z$

2.4.1 The position vector

In $CS_2$ \( \mathbf{r}_2^A = \rho \mathbf{i}_2 + z \mathbf{k}_2 \)

Thus \( \mathbf{r}_1^A = C_{21} \cdot \mathbf{r}_2^A \) and since

\[
C_{21} = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{r}_1^A = C_{21} \cdot \mathbf{r}_2^A = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\rho \\
0 \\
z
\end{bmatrix} = \begin{bmatrix}
\rho \cos \varphi \\
\rho \sin \varphi \\
z
\end{bmatrix}
\]

in vector form: \( \mathbf{r}_1^A = \rho \cos \varphi \mathbf{i}_1 + \rho \sin \varphi \mathbf{j}_1 + z \mathbf{k}_1 \)

2.4.2 The velocity

Expressed in vector form \( \mathbf{v}_2^A = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\rho \mathbf{i}_2 + z \mathbf{k}_2) = \dot{\rho} \mathbf{i}_2 + \rho \mathbf{j}_2 + \dot{z} \mathbf{k}_2 \)

where unit vector $\mathbf{k}_2$ remains constant (magnitude as well as direction does not change with time), therefore $\mathbf{k}_2 = 0$

The unit vector $\mathbf{i}_2$ rotates in the plane $x, y$ around the origin thus the velocity is given as

\[
\mathbf{v}_2^A = \frac{d\rho}{dt} \mathbf{i}_2 + \rho \frac{d\mathbf{j}_2}{dt} + \frac{dz}{dt} \mathbf{k}_2
\]

where

\[
\frac{d\mathbf{j}_2}{dt} = \frac{d\varphi}{dt} \times \mathbf{i}_2 = \mathbf{\omega}_2 \times \mathbf{i}_2 = \mathbf{j}_2 \cdot \mathbf{\omega}
\]

represents the transverse vector perpendicular to unit vector $\mathbf{i}_2$.

Thus the velocity \( \mathbf{v}_2^A = \frac{d\rho}{dt} \mathbf{i}_2 + \rho \frac{d\varphi}{dt} \mathbf{j}_2 + \frac{dz}{dt} \mathbf{k}_2 = \dot{\rho} \mathbf{i}_2 + \varphi \cdot \rho \mathbf{j}_2 + \dot{z} \mathbf{k}_2 \)

Where $\dot{\rho} \cdot \mathbf{i}_2 = \mathbf{v}_\rho$ represents the radial component of velocity

$\varphi \cdot \mathbf{j}_2 = \mathbf{v}_\varphi$ represents the transverse component of velocity

$\dot{z} \cdot \mathbf{k}_2 = \mathbf{v}_z$ represents the z-component of velocity

in matrix form: the transpose velocity in $CS_2$ \( \mathbf{v}_2^T = [\mathbf{v}_\rho \quad \mathbf{v}_\varphi \quad \mathbf{v}_z] \)

\[
\mathbf{v}_1^A = C_{21} \cdot \mathbf{v}_2^A = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\dot{\rho} \\
\rho \varphi \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
\dot{\rho} \cos \varphi - \rho \omega \sin \varphi \\
\dot{\rho} \sin \varphi + \rho \omega \cos \varphi \\
\dot{z}
\end{bmatrix}
\]
2.4.3 The acceleration

in vector form \( \mathbf{a}_2^A = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{\ddot{r}} + \mathbf{\dot{\rho}} \cdot \mathbf{j}_2 + \mathbf{\ddot{z}} \cdot \mathbf{k}_2) = \frac{d}{dt}(\dot{\mathbf{r}} + \dot{\mathbf{\rho}} \cdot \mathbf{j}_2 + \dot{\mathbf{z}} \cdot \mathbf{k}_2) \)

\[
\mathbf{a}_2^A = \mathbf{\ddot{r}} + \dot{\mathbf{\rho}} \cdot \mathbf{j}_2 + \dot{\mathbf{\rho}} \cdot \mathbf{\ddot{z}} \cdot \mathbf{k}_2 + \mathbf{\ddot{\rho}} \cdot \mathbf{j}_2 + \dot{\mathbf{\rho}} \cdot \mathbf{\ddot{z}} \cdot \mathbf{k}_2 + \mathbf{\ddot{z}} \cdot \mathbf{k}_2
\]

\[
\mathbf{a}_2^A = \mathbf{\ddot{r}} + \dot{\mathbf{\rho}} \cdot \mathbf{\omega} \cdot \mathbf{j}_2 + \dot{\mathbf{\rho}} \cdot \mathbf{\omega} \cdot \mathbf{j}_2 + \dot{\mathbf{\rho}} \cdot \mathbf{\alpha} \cdot \mathbf{j}_2 + \dot{\mathbf{\rho}} \cdot \mathbf{\omega} \cdot \mathbf{j}_2 + \mathbf{\ddot{z}} \cdot \mathbf{k}_2
\]

where: \( \frac{d\mathbf{j}_2}{dt} = \mathbf{\omega} \times \mathbf{j}_2 = -\mathbf{i}_2 \cdot \mathbf{\omega} \)

Thus giving acceleration \( \mathbf{a}_2^A = \mathbf{\ddot{r}} + 2\dot{\mathbf{\rho}} \cdot \mathbf{\dot{\phi}} \cdot \mathbf{j}_2 + \dot{\mathbf{\rho}} \cdot \mathbf{\omega} \cdot \mathbf{j}_2 + \mathbf{\ddot{z}} \cdot \mathbf{k}_2 \)

\[
(\mathbf{\dot{\rho}} - \mathbf{\rho} \mathbf{\ddot{\phi}}^2)\mathbf{i}_2 + (\dot{\mathbf{\rho}} \cdot \mathbf{\dot{\phi}} + \mathbf{\rho} \cdot \mathbf{\dot{\phi}}^2)\mathbf{j}_2 + \mathbf{\ddot{z}} \cdot \mathbf{k}_2
\]

where \( (\mathbf{\dot{\rho}} - \mathbf{\rho} \mathbf{\ddot{\phi}}^2) \) represents the radial acceleration

\( (\dot{\mathbf{\rho}} \mathbf{\dot{\phi}} + \mathbf{\rho} \mathbf{\ddot{\phi}}) \) represents the transverse acceleration

\( \mathbf{\ddot{z}} \) represents the acceleration in the z-axis direction

in matrix form:

\[
\mathbf{a}_2^{AT} = \begin{bmatrix}
\mathbf{\ddot{r}} - \mathbf{\rho} \mathbf{\ddot{\phi}}^2 & \mathbf{\dot{\rho}} \cdot \mathbf{\dot{\phi}} + 2\mathbf{\dot{\rho}} \cdot \mathbf{\dot{\phi}} & \mathbf{\ddot{z}}
\end{bmatrix}
\]

\[
\mathbf{a}_1^A = \mathbf{C}_2^A \mathbf{a}_2^A = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0
\sin \varphi & \cos \varphi & 0
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{r}} - \mathbf{\rho} \mathbf{\ddot{\phi}}^2 \\
\mathbf{\dot{\rho}} \mathbf{\dot{\phi}} + 2\mathbf{\dot{\rho}} \cdot \mathbf{\dot{\phi}} \\
\mathbf{\ddot{z}}
\end{bmatrix}
\]

In this presentation we associated angle \( \varphi \) with angular vector coordinate \( \varphi = \varphi \cdot \mathbf{k}_1 = \varphi \cdot \mathbf{k}_2 \)

thus angular velocity \( \dot{\varphi} = \dot{\varphi} \cdot \mathbf{k}_1 = \mathbf{\omega} \cdot \mathbf{k}_1 = \mathbf{\omega} \)

and angular acceleration \( \ddot{\varphi} = \ddot{\varphi} \cdot \mathbf{k}_1 = \ddot{\mathbf{\omega}} \cdot \mathbf{k}_1 = \mathbf{\alpha} \)

2.4.4 Special cases

a) \( z = 0, \mathbf{\rho} = \text{const.}, \dot{\varphi} \dot{\phi} \)

The particle (the point A) is restricted to the plane \( x,y \) only and moves in a way that the trajectory of the point A is a circle in the plane \( x,y \).

The velocity and acceleration expressed in general coordinates in the previous paragraph is:

\[
\mathbf{v}_2^A = \mathbf{\dot{r}} + \dot{\mathbf{\rho}} \cdot \mathbf{j}_2 + \dot{\mathbf{z}} \cdot \mathbf{k}_2
\]

\[
\mathbf{a}_2^A = (\mathbf{\ddot{r}} - \mathbf{\rho} \mathbf{\ddot{\phi}}^2)\mathbf{i}_2 + (\dot{\mathbf{\rho}} \cdot \mathbf{\dot{\phi}} + \mathbf{\rho} \cdot \mathbf{\dot{\phi}}^2)\mathbf{j}_2 + \mathbf{\ddot{z}} \cdot \mathbf{k}_2
\]

Thus for our particular case in CS2: \( \mathbf{v}_2^A = \dot{\mathbf{\varphi}} \cdot \mathbf{\rho} \cdot \mathbf{j}_2 \)

in CS1: \( \mathbf{v}_1^A = -\mathbf{\varphi} \sin \varphi \cdot \mathbf{\rho} \cdot \mathbf{i}_1 + \mathbf{\rho} \cdot \mathbf{\cos \varphi \cdot j}_1 \)
and acceleration in CS2: \( \mathbf{a}_2^A = -\rho \cdot \dot{\phi}^2 \mathbf{i}_2 + \rho \cdot \dot{\phi} \cdot \mathbf{j}_2 \)

where \( \mathbf{a}_2^A = -\rho \cdot \dot{\phi}^2 \mathbf{i}_2 \) represents the normal acceleration oriented always towards the centre of curvature of the trajectory
\( \mathbf{a}_2^A = \rho \cdot \dot{\phi} \cdot \mathbf{j}_2 \) represents the tangential component of an acceleration

in CS1: \( \mathbf{a}_1^A = (-\rho \cdot \dot{\phi}^2 \cos \varphi - \rho \cdot \dot{\phi} \sin \varphi) \mathbf{i}_1 + (\rho \cdot \dot{\phi} \cos \varphi - \rho \cdot \dot{\phi}^2 \sin \varphi) \mathbf{j}_1 \)

where \(-\rho \cdot \dot{\phi}^2 \cos \varphi = a_{nx} \) is the x-component of the normal acceleration in CS1
\(-\rho \cdot \dot{\phi} \sin \varphi = a_{rx} \) is the x-component of the tangential acceleration in CS1
\(\rho \cdot \dot{\phi} \cos \varphi = a_{ry} \) is the y-component of the tangential acceleration in CS1
\(-\rho \cdot \dot{\phi}^2 \sin \varphi = a_{ny} \) is the y-component of the normal acceleration in CS1

b) \( z=0, \rho(\rho, \varphi(t)) \)
The particle (point) moves in the plane \( x, y \) and description of the problem uses polar coordinates \((\rho, \varphi)\)

2.5 PARTICLE TRAJECTORY
The motion could be classified with respect to the trajectory as:

2.5.1 Rectilinear motion
The position of a point is described as a function of curvilinear coordinates \( s : \mathbf{r} = \mathbf{r}(s) \)
where
\[
\frac{d\mathbf{r}}{ds} = \tau \quad \text{and} \quad \tau^2 = 1
\]
The necessary condition for rectilinear motion is given as: \( \tau = \text{const.} \)
The velocity is given as:
\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} = \frac{d\mathbf{r}}{ds} = \tau \cdot \dot{s}
\]
and acceleration:
\[
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\tau \cdot \dot{s}) = \tau \cdot \ddot{s}
\]
If: \( \dot{s} = \text{const.} \) therefore \( \ddot{s} = 0 \) …uniform rectilinear motion
\( \dot{s} \neq \text{const.} \) thus \( \ddot{s} \neq 0 \) …accelerated rectilinear motion
for \( \ddot{s} = \text{const.} \) then uniformly accelerated rectilinear motion
for \( \ddot{s} \neq \text{const.} \) then non-uniformly accelerated rectilinear motion
2.5.2 Curvilinear motion

In case of curvilinear motion \( \mathbf{r} = \mathbf{r}(s) \) and \( s = s(t) \) and velocity is expressed as:

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \cdot \frac{ds}{ds} = \frac{dr}{ds} \cdot \frac{ds}{dt} = \mathbf{\tau} \cdot \dot{s}
\]

Where \( \mathbf{\tau}^2 = 1 \) as well as \( \mathbf{\tau} \neq \text{const.} \) since the unit vector changes its direction.

The acceleration is:

\[
a = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} (\mathbf{\tau} \cdot \dot{s}) = \mathbf{\tau} \cdot \ddot{s} + \mathbf{n} \cdot \dot{s}^2 \cdot \frac{1}{\rho} = \mathbf{a}_r + \mathbf{a}_n
\]

In case that:

a) \( \dot{s} = \text{const.} \) then \( \ddot{s} = 0 \)

therefore \( \mathbf{a} = \mathbf{n} \frac{1}{\rho} \dot{s}^2 = \mathbf{a}_n \) and the point moves along the circular trajectory with uniform velocity.

b) \( \dot{s} \neq \text{const.} \) then \( \ddot{s} \neq 0 \) and motion is non-uniformly accelerated or

\( \ddot{s} = \text{const.} \) where \( \mathbf{a} = \mathbf{\tau} \ddot{s} + \mathbf{n} \frac{\dot{s}^2}{\rho} \) motion is uniformly accelerated.

2.6 Harmonic motion

The motion of a point (particle) that is described by equation \( x = A \sin(\omega \cdot t + \varphi) \)

where \( A \) represents the amplitude (max. deviation from neutral position) [m]
\( \omega \) represents the angular frequency [s\(^{-1}\)]
\( \varphi \) represents the phase shift [rad]
\( x \) represents the instant distance of the particle

is called the \textbf{harmonic motion}

Velocity is in this case is given as: \( \mathbf{v} = \frac{dx}{dt} = A \cdot \omega \cos(\omega \cdot t + \varphi) \) and

Acceleration is given by: \( \mathbf{a} = \frac{dv}{dt} = -A \cdot \omega^2 \sin(\omega \cdot t + \varphi) \)

For the initial condition: \( t = 0, \ x = x_0, \ v = v_0, \ a = a_0 \)

the kinematical quantities are:

\( x_0 = A \sin \varphi, \ \ v_0 = A \omega \cos \varphi, \ \ a_0 = -A \omega^2 \sin \varphi \)
Graphical interpretation of harmonic motion can be represented as the rectification of all kinematical quantities in time

\[ \omega T = 2\pi \]

Where \( T \) represents the period \( T = \frac{2\pi}{\omega} \) [s] thus frequency \( f = \frac{1}{T} \) [Hz]

Amplitude of the motion can be expressed from \( x_0 = A \sin \varphi \) and \( v_0 = A \omega \cos \varphi \)

\[
A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \quad \text{and} \quad \tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \frac{x_0 \omega}{v_0}
\]

Thus the kinematical quantities can be expressed as function of rotating vector \( r_x, r_v, r_a \)

Where

\[ |r_x| = A; \]
\[ |r_v| = A \omega; \]
\[ |r_a| = A \omega^2 \]
2.6.1 Composition of harmonic motions in the same direction

a) If $\omega_1 = \omega_2 = \ldots = \omega_n = \omega$ then

$$x_1 = A_1 \sin(\omega t + \phi_1) = A_1 e^{i(\omega t + \phi_1)}$$
$$x_2 = A_2 \sin(\omega t + \phi_2) = A_2 e^{i(\omega t + \phi_2)}$$
$$\vdots$$
$$x_n = A_n \sin(\omega t + \phi_n) = A_n e^{i(\omega t + \phi_n)}$$

The resulting motion is again harmonic motion described as:

$$x = \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} A_j \sin(\omega \cdot t + \phi_j) = \sum_{j=1}^{n} A_j e^{i(\omega t + \phi_j)} = e^{i\omega t} \left( \sum_{j=1}^{n} A_j e^{i\phi_j} \right)$$

thus

$$x = A_v e^{i(\omega t + \phi_v)}$$

substituting for $t = 0$ we get:

$$x = A_v e^{i\phi_v}$$

from where we get the final amplitude and phase shift

$$A_v = \sqrt{\left( \sum_{j=1}^{n} A_j \cos \phi_j \right)^2 + \left( \sum_{j=1}^{n} A_j \sin \phi_j \right)^2}$$

and

$$\phi_v = \frac{\sum_{j=1}^{n} A_j \sin \phi_j}{\sum_{j=1}^{n} A_j \cos \phi_j}$$

b) If $\omega_1 \neq \omega_2 \neq \ldots \neq \omega_n$ then each motion is described by its own equation

$$x_1 = A_1 \sin(\omega_1 \cdot t + \phi_1) = A_1 e^{i(\omega_1 t + \phi_1)}$$
$$x_2 = A_2 \sin(\omega_2 \cdot t + \phi_2) = A_2 e^{i(\omega_2 t + \phi_2)}$$
$$\vdots$$
$$x_n = A_n \sin(\omega_n \cdot t + \phi_n) = A_n e^{i(\omega_n t + \phi_n)}$$

and the final motion is described by equation:

$$x = \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} A_j \sin(\omega_j t + \phi_j) = \sum_{j=1}^{n} A_j e^{i(\omega_j t + \phi_j)}$$

The final motion composed from harmonic motions with different angular frequencies is not a harmonic motion, since the resulting amplitude is not constant.

In case that $\omega_1 = \frac{2\pi n_1}{T}$ and $\omega_2 = \frac{2\pi n_2}{T}$ and at the same time the ratio $\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}$ is a rational number the resulting motion is said to be periodic motion.
2.6.2 Composition of two perpendicular harmonic motions

The motion of two particles moving in two perpendicular directions is defined by equations:

\[ x = A_1 \sin(\omega_1 t + \varphi_1) = A_1 e^{i(\omega_1 t + \varphi_1)} \quad \text{and} \]
\[ y = A_2 \sin(\omega_2 t + \varphi_2) = A_2 e^{i(\omega_2 t + \varphi_2)} \]

These equations define curves known as Lissajous picture. The solution is quite demanding and beyond our scope. The relatively simple solutions exists for special cases, when \( \omega_1 = \omega_2 \) and assumption \( A_1 = A_2 \) leading to equation of ellipse on conjugate axes.

2.7 MOTION OF A SET OF PARTICLES

Set of particles can be either connected set or particles or number of two or more unconnected particles moving in the same reference system. Thus the relationship between particles has to be taken into consideration. Let’s take the case of particle A and B as shown on diagram:

Both particles are connected via inextensible cable carried over the pulleys. This imposes non-penetrable condition between them:

\[ l = s_A + h + 2(h - s_B) \]

The additional length of the cable between the upper datum and the ceiling as well as the portion of the cable embracing the pulleys will remain constant during the motion thus does not play any role in the kinematical description.

Investigating mobility of the set of particles would define the number of independent coordinates that in our case is \( i = 1 \)

The path of a particle A is not identical with path of the particle B and the relation between them has to be described based on the joints involved. Thus except of the non-penetration condition the support at A has to be taken into consideration as well as the supports for the pulleys and body B.

Having the basic condition of the inextensible length we can evaluate the relation between velocities of the particle A and B as a time derivative of the \( l \). Thus

\[ 0 = v_A + 2 \cdot v_B \]

Then we can conclude that for motion of the particle A in positive direction (away from the datum in the direction of \( s_A \)) the particle B will move upwards with velocity \( v_B = \frac{v_A}{2} \).
3 SOLID BODY MOTION

As we announced before the model body adopted in kinematics is again non-deformable, therefore the distance between two points A, B on a solid body will follow the rule mathematically expressed as: \( AB = \text{const} \).

3.1 TRANSLATION MOTION OF A SOLID BODY

The position and trajectory of two points A, B is investigated. If two moving points will draw their trajectory in two parallel planes, thus their trajectories are parallel curves. The change of position from point A to A’ and B to B’ is described by vectors \( \mathbf{p} \) that are parallel. The advance motion is rectilinear (solid line) if vector \( \mathbf{p} \) is a straight line and curvilinear if it is a curve (dotted line).

The change of position of the point A is
\[
\mathbf{r}_A = \mathbf{r}_A + \mathbf{p}
\]
of the point B
\[
\mathbf{r}_B = \mathbf{r}_B + \mathbf{p}
\]

Thus
\[
\mathbf{r}_A - \mathbf{r}_A = \mathbf{r}_B - \mathbf{r}_B \quad \text{or rearranged} \quad \mathbf{r}_B - \mathbf{r}_A = \mathbf{r}_B - \mathbf{r}_A
\]

Expressing vectors \( \mathbf{r}_B \) and \( \mathbf{r}_B' \) with respect to the reference point A/A’ we can prove previous statement about parallel vectors \( \mathbf{r}_{BA} = \mathbf{r}_{B'A'} = \text{const} \).

3.1.1 Investigating kinematic quantities

Position

Point A is the reference point attached to the body associated with moving CS2 \((x_2, y_2, z_2)\). Position of a point B is given as
\[
\mathbf{r}_B^B = \mathbf{r}_A^A + \mathbf{r}_{BA}^A
\]
and
\[
\mathbf{r}_1^B = x_2^{BA} \cos \alpha_1 + y_2^{BA} \cos \alpha_2 + z_2^{BA} \cos \alpha_3
\]
thus the position of the point B in CS₁ is

\[ x_1^B = x_1^A + x_2^{BA} \cos \alpha_1 + y_2^{BA} \cos \alpha_2 + z_2^{BA} \cos \alpha_3 \]
\[ y_1^B = y_1^A + x_2^{BA} \cos \beta_1 + y_2^{BA} \cos \beta_2 + z_2^{BA} \cos \beta_3 \]
\[ z_1^B = z_1^A + x_2^{BA} \cos \gamma_1 + y_2^{BA} \cos \gamma_2 + z_2^{BA} \cos \gamma_3 \]

or in matrix form \( r_1^B = r_1^A + C_{21} \cdot r_2^{BA} \)

Transformation matrix \( C_{21} \) contains the cosines of all angles among axes of coordinate system. Since all vectors remain parallel vectors the angles between particular axes are constant. Therefore \( C_{21} = const. \)

**Velocity**

In vector form the velocity of point B is given as:

\[ v_1^B = \frac{d}{dt} r_1^B = \frac{d}{dt} (r_1^A + r_1^{BA}) = v_1^A + v_1^{BA} = v_1^A \] since the \( r_1^{BA} = const. \)

In matrix form: \( v_1^B = \dot{r}_1^B = \dot{r}_1^A + \dot{C}_{21} \cdot r_2^{BA} + C_{21} \cdot \dot{r}_2^{BA} \)

and \( \dot{C}_{21} = 0 \) since \( C_{21} = const. \)

Thus the final matrix form is: \( v_1^B = \dot{r}_1^B = \dot{r}_1^A = v_1^A \)

**Acceleration**

In vector form the acceleration of point B is given as:

\[ a_1^B = \frac{d}{dt} v_1^B = \frac{d}{dt} (v_1^A + v_1^{BA}) = v_1^A = a_1^A \] since \( v_1^{BA} = 0 \)

In matrix form: \( a_1^B = \ddot{r}_1^B = \ddot{r}_1^A = v_1^A = a_1^A \)

### 3.2 ROTATION OF A SOLID BODY AROUND FIXED AXIS

If two points of a moving solid body are stationary then

\[ \ddot{v}_{O_1} = \ddot{v}_{O_2} = 0 \]

then the solid body rotates around axis that passes through these points \( O_1 \) and \( O_2 \). The positional vectors describe their position in CS₁ by

\( r_{O1} = const., r_{O2} = const. \)
Any point on the line specified by points \(O_1\) and \(O_2\) can be described as
\[
\mathbf{r}_{O_1} = \mathbf{r}_{O_2} + \lambda \mathbf{r}_{O_1 O_2}
\]
and the velocity of point \(O_3\) is obtained by the first derivative of its position, thus
\[
\mathbf{r}_{O_1} = \mathbf{v}_{O_1} = 0
\]

Conclusion: There are infinity of points, laying on the line specified by points \(O_1\) and \(O_2\), which have a zero velocity.
The loci of all points that have a zero velocity is called the \textit{axis of rotation}.

The path of a point \(A\) that lays in the plane \((x, z)\) rotating around the axis \(z\), is a circle with radius \(\rho\), which corresponds to the projection of the position vector \(\mathbf{r}_A\) into the \(x, y\) plane.
The instant position of a point \(A\) depends on the instant angle of rotation \(\varphi = \varphi(t)\) further on we assign to this angular coordinate a vector quantity that follows the right hand rule.

The angular velocity that describes the rate of change of angular coordinate is expressed as the average value of angular velocity:
\[
\omega_{\text{avg}} = \frac{\Delta \varphi}{\Delta t}
\]
The instant angular velocity is given as:
\[
\lim_{\Delta t \to 0} \frac{\Delta \varphi}{\Delta t} = \frac{d\varphi}{dt} = \omega\quad \text{thus} \quad \omega = \varphi = \mathbf{e}_\varphi \varphi
\]

In the same way we can express the angular acceleration.
The average acceleration is given as:
\[
\mathbf{a}_{\text{avg}} = \frac{\Delta \omega}{\Delta t}
\]
Instant acceleration is
\[
\lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} = \mathbf{a}
\]
thus
\[
\mathbf{a} = \dot{\omega} = \mathbf{e}_\varphi \dot{\varphi}
\]

If the solid body rotates around fixed axis then all point of the body have the same angular velocity and acceleration.

\textbf{3.2.1 Finding the velocity of an arbitrary point}
The point \(B\) is attached to the rotating solid body
Then the \textit{position of a point \(B\)} in CS2 is given as
\[
\mathbf{r}_2^B = x_2^B \mathbf{i}_2 + y_2^B \mathbf{j}_2 + z_2^B \mathbf{k}_2
\]
or with respect to the point $O$ around which the point $B$ rotates with radius $\rho$

$$\mathbf{r}_2^B = \mathbf{r}_2^O + \rho$$

Point $B$ moves on the circular trajectory in time $\Delta t \to 0$ a distance

$$dr = d\varphi \cdot \rho = d\varphi \cdot r_B \cdot \sin \beta$$

in vector form:

$$d\mathbf{r} = d\varphi \times \mathbf{r}_B = d\varphi \times \rho$$

since vectors $\mathbf{r}$ and $\rho$ lay in the same plane and together with vector $\varphi$ form the plane to which the path increment $d\mathbf{r}$ is orthogonal (perpendicular).

Thus the velocity of the point $B$ is

$$\mathbf{v}_B = \frac{d\mathbf{r}}{dt} = \frac{d\varphi}{dt} \times \mathbf{r}_B = \frac{d\varphi}{dt} \times \rho = \varphi \times \mathbf{r}_B = \varphi \times \rho$$

the module of velocity:

$$|\mathbf{v}_B| = \omega \cdot r_B \cdot \sin \beta = \omega \cdot \rho$$

or in vector from:

$$\mathbf{v}_B = \varphi \times \mathbf{r}_B = \begin{pmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{pmatrix} = i(\omega_x r_z - \omega_z r_x) + j(\omega_y r_z - \omega_z r_y) + k(\omega_x r_y - \omega_y r_x)$$

then

$$\mathbf{v}_B = i\mathbf{v}_x + j\mathbf{v}_y + k\mathbf{v}_z$$

with module

$$|\mathbf{v}_B| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The orientation of the linear velocity is given by right hand rule: Grabbing the axis of rotation with our right hand in a way that the thumb points in the direction of angular velocity then fingers would show the direction of velocity of the particular point of a body.

If the position of the point $B$ is expressed in matrix form

$$\mathbf{r}_1^B = C_{21} \cdot \mathbf{r}_2^B$$

then the velocity is the first derivative of position, thus

$$\mathbf{v}_1^B = \dot{\mathbf{r}}_1^B = \frac{d}{dt}(C_{21} \cdot \mathbf{r}_2^B) = C_{21} \mathbf{r}_2^B + C_{21} \dot{\mathbf{r}}_2^B$$

Since the point $B$ is attached to and rotates with $CS_2$ then

$$\mathbf{r}_2^B = \text{const. and its first derivative is equal to zero.}$$

The velocity of the point $B$ is given as

$$\mathbf{v}_1^B = \dot{\mathbf{r}}_1^B = \dot{C}_{21} \mathbf{r}_2^B = \Omega_1 \cdot C_{21} \cdot \mathbf{r}_2^B$$

The first derivative of transformation matrix

$$\dot{C}_{21} = \Omega_1 C_{21} = C_{21} \cdot \Omega_2$$

and finally

$$\mathbf{v}_1^B = \Omega_1 \cdot C_{21} \cdot \mathbf{r}_2^B = C_{21} \cdot \Omega_2 \cdot \mathbf{r}_2^B$$

### 3.2.2 Finding the acceleration of an arbitrary point B

In vector form:

$$\mathbf{a}_B = \ddot{\mathbf{r}}_B = \frac{d}{dt}(\varphi \times \mathbf{r}_B) = \alpha \times \mathbf{r}_B + \varphi \times \mathbf{v}_B$$

and simultaneously

$$\mathbf{a}_B = \frac{d}{dt}(\varphi \times \rho) = \alpha \times \rho + \varphi \times \mathbf{v}_B$$

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Since the point B moves on the circular path the acceleration will have two components

**Tangential component**

\[
a^t_B = a \times \rho = a \times r_B = \begin{bmatrix} i & j & k \end{bmatrix} \begin{bmatrix} \alpha_x & \alpha_y & \alpha_z \\ \frac{r_x}{r_z} & \frac{r_y}{r_z} & \frac{r_z}{r_z} \end{bmatrix} = i\left(\alpha_x r_z - \alpha_z r_x + k\alpha_z r_y - \alpha_y r_z\right)
\]

and normal acceleration

\[
a^n_B = \omega \times v_B = \begin{bmatrix} i & j & k \end{bmatrix} \begin{bmatrix} \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{bmatrix} = i\left(\omega_x v_z - \omega_z v_x + k\omega_z v_y - \omega_y v_z\right)
\]

Their modules are:

\[
|a^t_B| = \sqrt{\left(a^t_x\right)^2 + \left(a^t_y\right)^2 + \left(a^t_z\right)^2} \quad \text{or} \quad |a^t_B| = \rho \omega \sin \beta = \alpha \rho
\]

\[
|a^n_B| = \sqrt{\left(a^n_x\right)^2 + \left(a^n_y\right)^2 + \left(a^n_z\right)^2} \quad \text{or} \quad |a^n_B| = \omega v_B = \omega^2 \rho = \omega^2 r_B = \frac{v_B^2}{\rho}
\]

In matrix form:  
\[
a^t_B = \ddot{\mathbf{v}}_B - \dot{\mathbf{r}}_B = \dot{\mathbf{C}}_{21} \cdot \mathbf{r}^B_2 + 2\dot{\mathbf{C}}_{21} \cdot \dot{\mathbf{r}}^B_2 + \mathbf{C}_{21} \cdot \ddot{\mathbf{r}}^B_2
\]

which leads to  
\[
a^t_B = \ddot{\mathbf{v}}_B = \dot{\mathbf{r}}_B = \dot{\mathbf{C}}_{21} \cdot \mathbf{r}^B_2
\]

then the second derivative of transformation matrix is:

\[
\dot{\mathbf{C}}_{21} = \dot{\Omega}_1 \mathbf{C}_{21} + \Omega_1 \dot{\mathbf{C}}_{21}
\]

\[
\ddot{\mathbf{C}}_{21} = \ddot{\Omega}_1 \mathbf{C}_{21} + \frac{\Omega_1 \dot{\Omega}_1}{2} \mathbf{C}_{21} = \dot{\Omega}_1 \mathbf{C}_{21} + \frac{\Omega_1^2}{4} \mathbf{C}_{21}
\]

\[
\dddot{\mathbf{C}}_{21} = \dot{\mathbf{A}}_1 \mathbf{C}_{21} + \Omega_1^2 \mathbf{C}_{21}
\]

Thus the acceleration of the point B is  
\[
a^t_B = (\mathbf{A}_1 \mathbf{C}_{21} + \Omega_1^2 \mathbf{C}_{21}) \mathbf{r}^B_2 = (\mathbf{A}_1 + \Omega_1^2) \mathbf{C}_{21} \mathbf{r}^B_2 = (\mathbf{A}_1 + \Omega_1^2) \mathbf{r}^B_1
\]

where the first component represents the tangential acceleration

\[
a^t_1 = A_1 \mathbf{C}_{21} \mathbf{r}^B_2 = A_1 \mathbf{r}^B_1 = A_1 \rho \quad \text{where} \quad A_1 = \begin{bmatrix} 0 & -\alpha_z & \alpha_y \\ \alpha_z & 0 & -\alpha_x \\ -\alpha_y & \alpha_x & 0 \end{bmatrix}
\]

and the second component represents the normal acceleration

\[
a^n_1 = \Omega_1^2 \mathbf{C}_{21} \mathbf{r}^B_2 = \Omega_1^2 \mathbf{r}^B_1 = \Omega_1 \mathbf{v}_1^B
\]

The course of motion is recorded by the tangential acceleration  
\[
\alpha_t = \alpha \cdot \rho
\]

where  
\[
\alpha = \frac{d\omega}{dt} = \dot{\omega}
\]
There could be two situations:

a) \( \omega = \text{const.} \Rightarrow \alpha = 0 \)

Thus \( a^r = 0 \) and \( a^\theta = \rho \cdot \omega^2 = \frac{v^2}{\rho} \)

These characteristics represent uniform motion of the particle on the circle and the acceleration that occurs is the normal acceleration.

b) \( \omega \neq \text{const.} \Rightarrow \alpha \neq 0 \)

In this case the angular acceleration \( \alpha \) can become:

i) \( \alpha = \text{const.} \) thus \( \alpha_r = \alpha \cdot \rho = \text{const.} \) (assuming \( \rho = \text{const.} \))

These characteristics represent uniformly accelerated motion on the circular path – uniformly accelerated rotation.

ii) \( \alpha \neq \text{const.} \) thus \( \alpha_r \neq \alpha \cdot \rho \neq \text{const.} \)

These characteristics represent non-uniformly accelerated motion on the circle, non-uniformly accelerated rotation.

In both cases the normal acceleration will occur. \( a^\theta = \rho \cdot \omega^2 = \frac{v^2}{\rho} \)

3.2.3 Solid body kinematics consequences (the geometrical dependency)

Providing the graphical solution for kinematic quantities, we need to record velocity and acceleration in a graphical form. For this purpose the length and velocity scale has to be given, while the remaining scales are calculated.

Where \( l^B_\rho \) represents the length of the radius vector \( \rho \)

\[ l_\rho = \frac{\rho}{s_\nu} \text{ thus } \rho = s_\nu \cdot l_\rho \]

and velocity \( v = s_\nu \cdot l_\nu \)

Therefore the angle \( \alpha_\nu \), is given as

\[ \tan \alpha_\nu = \frac{l_\nu}{l_\rho} = \frac{v \cdot s_\nu}{\rho \cdot s_\nu} = \omega \cdot k_\nu \]

where \( k_\nu \) is a velocity scale constant.

Since all points on the body have the same angular velocity \( \omega \) we can conclude: Sentence about velocities:

All arrowheads of velocity vectors, for particular points of a rotating body, are visible from the fixed point of rotation \( (v=0) \) under the same angle \( \alpha_\nu \) at the instant moment.
**Acceleration – the graphical solution**

A tangential and normal components of acceleration has to be recorded.

*The normal component of acceleration:*

\[ a^n = \rho \cdot \omega^2 = \frac{\mathbf{v}^2}{\rho} \]

Thus from Euclid’s law about the height in the triangle follows the graphical construction of normal acceleration.

The acceleration scale has to be calculated!

\[
\text{Therefore } a_n = \frac{s^2 \cdot l^2}{s_i \cdot l \rho} = s_n \cdot l_{a_n} \quad \text{where } l_{a_n} = \frac{l^2}{l \rho} \quad \text{and } \quad s_n = \frac{s^2}{s_i}
\]

*The tangential component of acceleration:*

\[ \alpha_t = \alpha \cdot \rho \]

The direction of tangential component corresponds to the direction of velocity hence further analogy with velocity is obvious.

\[
\tan \alpha_{a_t} = \frac{l_{a_t}}{l \rho} = \frac{a^\tau \cdot s_i}{\rho \cdot s_n} = \alpha \cdot k_{a_t},
\]

where \( k_{a_t} \) is a tangential acceleration scale constant.

Since all points on the body have the same angular acceleration \( \alpha \) we can conclude:

**Sentence about tangential acceleration:**

All arrowheads of tangential acceleration vectors, at the points on the rotating body, are at the instant visible under the same angle \( \alpha_t \) from the fixed point of rotation (\( v=0 \)).

The total acceleration is given as a sum of its components \( \mathbf{a} = \mathbf{a}^\tau + \mathbf{a}^n \)

\[
\tan \beta = \frac{\mathbf{a}^\tau}{a^n} = \frac{\alpha \rho}{\omega^2 \rho} = \frac{\alpha^2}{\omega^2} = k_{a_t}
\]

where \( k_a \) represents the scale constant of total acceleration.

Since \( \alpha \) and \( \omega \) are constant for all points on the body we conclude:

**The total acceleration of the point on the rotating solid body makes an angle \( \beta \) from its normal to the trajectory that remains constant for all points on the body.**

And finally we can finalise based on the background:

\[
\tan \alpha_a = \frac{l_{a_t}}{l^2 - l_{a_t}^2} = \frac{\alpha}{m_a m_i^1 - \omega^2}.
\]

Together with the angular kinematic quantities \( \alpha, \omega \) that are the same for all points of the body:

**All arrowheads of total acceleration vectors of all points on the rotating body, are at the instant visible under the same angle \( \alpha_a \) from the fixed point of rotation (\( v=0 \)).**
3.3 UNIVERSAL PLANAR MOTION

If all points of a body move in planes parallel to the fixed (stationary) basic plane then we say that the body moves in planar motion.

If the trajectories of all points that lay on the line perpendicular to that plane are planar curves then the motion is a universal planar motion.

The initial position of the body is described via points A, B. A positional vector for point B using a reference point A describes the initial position of a body:

\[ \mathbf{r}^B = \mathbf{r}^A + \mathbf{r}^{BA} \]

During the time interval \( t \) their position changes to a new location \( A_1 \) and \( B_1 \) thus

\[ \mathbf{r}^{A_1} = \mathbf{r}^A + \mathbf{r}^{BA_1} \]

As seen on the diagram

\[ |\mathbf{r}^{BA}| \neq |\mathbf{r}^{B,A}| \]

means that the vector changes its orientation but not its magnitude.

Therefore we can imagine the universal planar motion as a sequence of translation motion followed by rotation that could be expressed in a short way as:  GPM = TM + RM

**Note:** Both motion are happening in the same time and this approach is just imaginary.

### 3.3.1 The position

of the point B can be expressed in vector or matrix form:

\[ \mathbf{r}_1^B = \mathbf{r}_1^A + \mathbf{r}_1^{BA} \]

with use of transformation matrix \( \mathbf{C}_{21} \):  

\[ \mathbf{r}_1^B = \mathbf{r}_1^A + \mathbf{C}_{21} \cdot \mathbf{r}_2^{BA} \]

that leads to two equations in vector form:

\[
\begin{align*}
x_1^B &= x_1^A + x_2^{BA} \cos \varphi - y_2^{BA} \sin \varphi \\
y_1^B &= y_1^A + x_2^{BA} \sin \varphi + y_2^{BA} \cos \varphi
\end{align*}
\]

or in matrix form:

\[
\begin{bmatrix}
x_1^B \\
y_1^B \\
z_1^B
\end{bmatrix} =
\begin{bmatrix}
x_1^A \\
y_1^A \\
z_1^A
\end{bmatrix} +
\begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2^{BA} \\
y_2^{BA} \\
z_2^{BA}
\end{bmatrix}
\]

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3.3.2 The velocity

of a point B can be expressed:

in vector form: \[ v_B^1 = \frac{d}{dt} (r_A^1 + r_{BA}) = v_A^1 + v_{BA}^1 \]

thus \[ v_B^1 = v_A^1 + \omega \times r_{BA}^1 \]

or in matrix form:
\[ v_B^1 = v_A^1 + v_{BA}^1 = v_A^1 + \hat{C}_{21} r_{BA}^1 \]
\[ v_B^1 = v_A^1 + \Omega_1 r_{CA}^1 = v_A^1 + \Omega_1 r_{BA}^1 \]

There is a rotation of the point B with respect to the point A thus we say that there is a relative motion of the point B around point A.

Thus \[ v_{BA}^1 = \omega \times r_{BA}^1 \]

where \( \omega \) represents the angular velocity of a relative motion with respect to the point A.

The relative angular velocity is constant for all points of the body thus: \[ \omega = \frac{v_{BA}^1}{BA} = \frac{v_{CA}^1}{CA} = \text{const.} \]

Then we can get the components of velocity:
\[
\begin{bmatrix}
    v_A^x \\
    v_A^y \\
    0
\end{bmatrix}
= \begin{bmatrix}
    v_B^x \\
    v_B^y \\
    0
\end{bmatrix}
+ \begin{bmatrix}
    0 & -\omega & 0 \\
    \omega & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_{BA}^x \\
    y_{BA}^y \\
    0
\end{bmatrix}
\]

Graphical solution:

The velocity of a point A is known, thus we need to find the velocity at the point B based on the vector equation:
\[ v_B^1 = v_A^1 + \omega \times r_{BA}^1 \]
3.3.3 The pole of motion

If \( \omega \neq 0 \) then there exists just one point on the body that has zero velocity at the instant and belongs to the moving body (or the plane attached to the moving body). This point is known as the instantaneous centre of rotation or the pole of motion.

The position of a pole \( P \) is given as:

\[
\mathbf{r}_P^P = \mathbf{r}_A + \mathbf{r}_{PA}
\]

in matrix form:

\[
\mathbf{r}_P^P = \mathbf{r}_A + \mathbf{r}_{PA} = \mathbf{r}_A + \mathbf{C}_{21} \mathbf{r}_{PA}^P
\]

Then the velocity of the pole is:

\[
\mathbf{v}_P^P = \mathbf{v}_A^P + \mathbf{v}_{PA}^P
\]

since the linear velocity at this location is zero, then

\[
0 = \mathbf{v}_A^P + \mathbf{v}_{PA}^P \rightarrow \mathbf{v}_{PA}^P = -\mathbf{v}_A^P
\]

For arbitrary reference point we would get similar answer:

\[
0 = \mathbf{v}_B^P + \mathbf{v}_{PB}^P \rightarrow \mathbf{v}_{PB}^P = -\mathbf{v}_B^P
\]

To find a position of the pole of a body moving with GPM we have to multiply the velocity equation by the angular velocity

Thus  

\[
0 = \mathbf{\omega} \times \mathbf{v}_A + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{PA})
\]

Rearranging the equation into a form

\[
0 = \mathbf{\omega} \times \mathbf{v}_A + \left[ \mathbf{\omega} \cdot (\mathbf{\omega} \times \mathbf{r}_{PA}) - \mathbf{r}_{PA} \cdot (\mathbf{\omega} \cdot \mathbf{\omega}) \right]
\]

while equating \( \mathbf{\omega} \cdot \mathbf{r}_{PA} = 0 \) and \( \mathbf{\omega} \cdot \mathbf{\omega} = \omega^2 \)

we get

\[
0 = \mathbf{\omega} \times \mathbf{v}_A + (-\omega^2 \mathbf{r}_{PA})
\]

from where we equate the pole positional vector

\[
\mathbf{r}_{PA}^P = \frac{\mathbf{\omega} \times \mathbf{v}_A^P}{\omega^2} = \frac{\mathbf{\omega} \times \mathbf{v}_A^P}{\omega^2} = \frac{1}{\omega^2} \begin{bmatrix} \mathbf{i}_1 & \mathbf{j}_1 & \mathbf{k}_1 \\ \mathbf{i}_x^A & \mathbf{j}_x^A & \mathbf{k}_x^A \\ \mathbf{i}_y^A & \mathbf{j}_y^A & \mathbf{k}_y^A \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \frac{1}{\omega^2} \begin{bmatrix} \mathbf{i}_1 (\omega \mathbf{v}_A^P) + \mathbf{j}_1 (\omega \mathbf{v}_A^P) \end{bmatrix}
\]
**Graphical solution:**

Based on $v_1^{PA} = -v_1^A$ and $v_1^{PB} = -v_1^B$.

The velocity $v^A$ is known and velocity at the point B is $v_1^B = v_1^A + v_1^{BA}$. 

thus if we know the velocity we know the direction of a normal to the trajectory.

*The pole of motion is at the intersection of normal $n_A$ and $n_B$.***

**Finding the velocity by means of the pole**

Position of a point B $r_1^B = r_1^P + r_1^{BP}$

or $r_1^B = r_1^P + C_2 r_2^{BP}$

Then the velocity is: $v_1^B = v_1^P + \Omega C_2 r_2^{BP}$

Since the velocity of the pole is zero we get $v_1^B = \Omega C_2 r_2^{BP}$

in matrix form

$\begin{bmatrix} v_x^B \\ v_y^B \\ 0 \end{bmatrix} = \begin{bmatrix} \omega & -\omega & 0 \\ \omega & 0 & \sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^{BP} \\ y^{BP} \\ 0 \end{bmatrix}$

that would result in

$v_{1x}^B = -\omega x_1^{BP}$ and $v_{1y}^B = \omega x_1^{BP}$

Thus giving the final velocity

$|v_1^B| = \sqrt{(v_{1x}^B)^2 + (v_{1y}^B)^2} = \sqrt{(-\omega x_1^{BP})^2 + (\omega x_1^{BP})^2} = \omega \sqrt{(y_1^{BP})^2 + (x_1^{BP})^2} = \omega x_1^{BP}$

Then the angular velocity is $\omega = \frac{v_1^B}{r_1^{BP}}$

From this result follows the interpretation for graphical solution:

$\omega = \frac{v_1^B}{r_1^{BP}} = \frac{s_i \cdot l_y}{s_i \cdot l_{ae}} = k_i \cdot \tan \alpha_i$,  

$\tan \alpha_i = \frac{1}{k_i}$

thus giving

\[\square\]
Conclusion:

All arrowheads of velocity vectors, for particular points of a rotating body, are visible from the instantaneous centre of rotation \( (v = 0) \) under the same angle \( \alpha \), at the instant moment.

3.3.4 Finding the acceleration

**Analytically**

We have already found the positional vector \( r_1^B \) and velocity \( v_1^B \)

\[
r_1^B = r_1^A + r_{BA}^1
\]

\[
v_1^B = v_1^A + v_{BA}^1
\]

where the velocity \( v_{BA}^1 \) represents the relative motion of the point B around the point A that can be expressed as

\[
v_{BA}^1 = \omega \times r_{BA}^1
\]

in vector form or

\[
v_{BA}^1 = \Omega_1 \cdot C_{21} \cdot r_{BA}^1 = \Omega_1 \cdot r_{BA}^1
\]

in matrix form.

Since we proved that \( \alpha = \vec{r} = \dot{v} \)

we can derive the equation of acceleration

\[
a_1^B = a_1^A + a_{BA}^1
\]

Where the acceleration of relative motion of a point B around point A will have two components since the relative motion is the rotation with a fixed point at A.

Thus

\[
a_{BA}^1 = \omega \times r_{BA}^1 + \omega \times v_{BA}^1
\]

in vector form. The acceleration in the matrix form is a result of derivation:

\[
r_1^B = r_1^A + r_{BA}^1 = r_1^A + C_{21} \cdot r_{2BA}^1
\]

\[
v_1^B = v_1^A + \Omega_1 \cdot C_{21} \cdot r_{2BA}^1
\]

\[
a_1^B = a_1^A + \dot{\Omega}_1 \cdot C_{21} \cdot r_{2BA}^1 + \Omega_1 \cdot \dot{C}_{21} \cdot r_{2BA}^1 + \Omega_1 \cdot C_{21} \cdot \dot{r}_{2BA}^1
\]

in case of solid body \( \dot{r}_{2BA}^1 = 0 \) since the distance between points A and B does not change, thus

\[
a_1^B = a_1^A + A_1 \cdot (C_{21} \cdot r_{2BA}^1 + \Omega_1 \cdot \Omega_1 \cdot C_{21} \cdot \dot{r}_{2BA}^1)
\]

\[
a_1^B = a_1^A + (A_1 + \Omega_1 \cdot \dot{r}_{2BA}^1) \cdot r_{BA}^1
\]

where

\[
A_1 = \begin{bmatrix}
0 & -\alpha_z & \alpha_y \\
\alpha_z & 0 & -\alpha_x \\
-\alpha_y & \alpha_x & 0
\end{bmatrix}
\]

represents the half symmetrical matrix of angular acceleration.
Graphically

To find the acceleration we will use the leading equation

\[ \mathbf{a}_i^B = \mathbf{a}_i^A + \mathbf{a}_i^{BA} \]

The acceleration of a point A is given and acceleration of the relative motion of point B is described by two components (tangential and normal) in the respective directions to the path of the point B.

The normal component of the acceleration \( \mathbf{a}_i^{BA} \) is found from the known velocity \( \mathbf{v}_i^{BA} \) (graphically by means of Euclid triangle).

Thus at the instant moment the angle \( \beta \) between the final acceleration of the relative motion and the normal to the path of relative motion is given by

\[ \tan \beta = \frac{a_i^{BA}}{a_i^{BA}_n} = \frac{\alpha}{\omega} = \text{const.} \]

Conclusion:

The final acceleration of the relative motion around point A makes an angle \( \beta \) with the normal component of the acceleration that is constant for all points of the body moving with relative motion.

In a similar way we can observe the tangential component of acceleration and the final acceleration. Thus this angle is given by:

\[ \tan \alpha = \frac{l_{v_{ni}}}{l_p - l_{a_{ni}}} = \frac{\alpha \cdot s_i \cdot s_i^2}{s_i^2 (s_i^2 - \omega^2 s_i^2)} \quad \text{where} \quad l_p = \frac{l_i^2}{l_{an}}. \]

Since \( \alpha \) and \( \omega \) are constant in the given time interval for all points of the rigid body then even the \( \tan \alpha_c = \text{const} \).

Conclusion:

The end points of tangent components of acceleration for all points on the body are seen from the centre of rotation under the same angle \( \alpha_c \) at the instant moment.
3.3.5 The instantenous centre of acceleration – the pole of acceleration

Similarly as for the pole of velocity P there is a pole of acceleration Q, the point that has zero acceleration at the instant.

The acceleration for this pole Q is given by:
\[
a_i^Q = a_i^A + a_{ir}^{QA} + a_{in}^{QA}
\]

Multiplying by \(\alpha\) from left
\[
0 = \mathbf{a} \times a_i^A + \mathbf{a} \times (\mathbf{a} \times r_i^{QA}) + \mathbf{a} \times [\mathbf{\omega} \times (\mathbf{\omega} \times r_i^{QA})]
\]

we receive
\[
0 = \mathbf{a} \times a_i^A + (-\alpha^2) r_i^{QA} + (-\omega^2)(\mathbf{a} \times r_i^{QA})
\]

and substituting for the expression
\[
\mathbf{a} \times r_i^{QA} = -a_i^A - (-\omega^2 r_i^{QA}) = -a_i^A + \omega^2 r_i^{QA}
\]

we find the positional vector of pole of acceleration
\[
r_i^{QA} = \frac{\mathbf{a} \times a_i^A + \omega^2 a_i^A}{\alpha^2 + \omega^4}
\]

Comparing this expression with the expression for pole of velocity
\[
r_i^{PA} = \frac{\mathbf{\omega} \times v_i^A}{\omega^2}
\]

shows that these two expressions are different, thus the two poles are different and we can conclude:

The pole of acceleration is not identical with pole of velocity.

If \(\omega \neq 0\) and \(\alpha \neq 0\) then there is one point on the moving plane that has acceleration \(a_i^Q = 0\).

This point lays at the intersection of lines that make an angle \(\beta\) with the directions of total acceleration of each and every point on the moving plane.
3.4 CENTRE OF THE TRAJECTORY CURVATURE

The body, moving with the universal planar motion, is defined by two points A and B and their trajectories $s_A$ and $s_B$ that the points are drawing in the plane. Thus the position of the pole of velocity that lies in the intersection of two normals can found at any instant.

The body is connected to the moving CS2 and the positions of all instantaneous poles of velocity are creating a curve $pH$ – the locus of all such positions is called the **moving polode** while the poles connected to the stationary fixed plane CS1 are creating a curve assigned the symbol $p_N$ – the **fixed polode**.

Thus we can imagine the universal planar motion as motion created by rolling of the locus of poles $p_N$ (associated with the fixed frame) on the locus of poles $p_H$ (associated with moving plane).

The pole velocity describes the rate of change of the pole position.

It is possible to find the rate of change of the positional vector using the reference point A with respect to the fixed frame

$$v^A = \omega \times a^A - \alpha \times v^A$$

And for the pole associated with the moving plane

$$v^p = \omega \times a^A - \alpha \times v^A$$

Since the derived equations of pole velocities are identical we can assign a symbol $v^\pi$ to be its pole of velocity and conclude:

**The point where both loci $p_N$ and $p_H$ touch at the instant is the pole of motion P (the instantaneous centre of rotation) and at this point the two curves have a common tangent $t_p$.**

**The pole velocity as the rate of change of the pole position will lie on the tangent $t_p$.**
The task is to investigate the velocity and acceleration of the point or define the velocity and acceleration of the whole body. The points of the moving body are drawing trajectories thus at the instant each point of the body is characterized by its normal to the motion and the radius of its trajectory.

While constructing the normal component of acceleration

\[ \frac{1}{\rho} \frac{1}{s^2} = a_n \]

the centre of curvature of the trajectory is needed to identify the radius \( \rho \).

To find the centre of curvature we can use different methods.

Only two methods are presented here:

a) analytical – Euler-Savary equation

The centre of curvature \( S^A \) associated with the point A on the moving body that draw the trajectory \( s_A \) is known. Thus we can find the pole velocity found by means of Hartman graphical method is used to show that

\[ \frac{v^A}{\rho_A} = \frac{v_\pi}{s} \Rightarrow \frac{r \omega}{r + s} = \frac{v_\pi \sin \vartheta}{s} \]

\[ \frac{\omega}{v_\pi} = \kappa = \frac{r + s}{rs} \sin \vartheta \Rightarrow \left( \frac{1}{s} + \frac{1}{r} \right) \sin \vartheta = \kappa \]

which represents the Euler-Savary equation used in analytical solution to find the centre of curvature.
b) Bobillier graphical method

The angle between the normal of a point and axis of collineation is the same as the angle measured between the normal of the other point and tangent to loci of pole positions in opposite direction.

There are two tasks:

1. The tangent \( t_p \) and a pair of conjugate points \( A, S_A \) are known and the centre of curvature of the trajectory of the point \( C \) has to be identified.

2. The two pairs of conjugate points \( A, S_A \), and \( B, S_B \) are known and the centre of curvature of the trajectory of the point \( C \) has to be identified.
3.5 COMBINED MOTION

The mechanism consisting of number of bodies undergo either planar or space motion that could be described as a combination of the relative motion between bodies and the driving/carrying motion of the actuator of the system with respect to the reference frame.

Analysing motion of each body in the mechanism:
- B1 – reference frame
- B2 –
- B3 –
- B4 –

Motion of the bodies attached to the frame is identified as a rotation with the fixed centre of rotation at O_{21} or O_{41} thus the body motion is defined by angular velocity and acceleration, \( \omega \) and \( \alpha \) respectively. In case of a simple motion such as rotation or translation we are able to identify the trajectory thus evaluate the kinematical quantities without any problem. In case of a planar or spatial motion of the body the trajectory is not a simple curve and thus there might be a problem to evaluate the kinematical quantities.

Therefore we introduce the strategy based on combined motion and implement imaginary split of complex motion into two motions: the motion of reference point and rotation around the reference point.

We can imagine

This could be recorded by a symbolic equation

\[
31 = 32 + 21
\]

And kinematical quantities can be expressed for identified point B as

Thus

\[
\mathbf{v}_{31} = \mathbf{v}_{32} + \mathbf{v}_{21}
\]

But

\[
\mathbf{a}_{31} = \mathbf{a}_{32} + \mathbf{a}_{21} + \mathbf{a}_{cor}
\]

To prove this statement, we have to define the velocity at the point B
3.5.1 Kinematical quantities by means of combined motion

The final motion of a point B symbolically:

\[ 3 \mathbf{1} = 3 \mathbf{2} + 2 \mathbf{1} \quad \text{as well as} \quad 3 \mathbf{1} = 3 \mathbf{4} + 4 \mathbf{1} \]

The positional vector of a point B is:

\[ \mathbf{r}_B^1 = \mathbf{r}_A^1 + \mathbf{r}_A^B = \mathbf{r}_1^A + (i_2 x_2^{BA} + j_2 y_2^{BA}) \]

3.5.2 The velocity of a point B:

\[ \mathbf{v}_B^1 = \mathbf{r}_B^1 = \mathbf{r}_A^1 + \mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA} \]

where

\[ \mathbf{i}_2 = \mathbf{\omega}_{21} \times \mathbf{i}_2 \]

\[ \mathbf{j}_2 = \mathbf{\omega}_{21} \times \mathbf{j}_2 \]

Substituting and rearranging we receive the equation of final velocity at the point B expressed in CSI

\[ \mathbf{v}_B^1 = \mathbf{v}_A^1 + \mathbf{\omega}_{21} \times (\mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA}) + \mathbf{i}_2 v^{BA}_{x_2} + \mathbf{j}_2 v^{BA}_{y_2} \]

where

the expression \( \mathbf{v}_A^1 + \mathbf{\omega}_{21} \times (\mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA}) = \mathbf{v}_{21}^B \) represents the velocity due to driving motion. Point B would move with driving velocity \( \mathbf{v}_{21}^B \) when virtually connected to the moving plane 2 (CS2)

the expression \( \mathbf{i}_2 v^{BA}_{x_2} + \mathbf{j}_2 v^{BA}_{y_2} = \mathbf{v}_{32}^B \) represents the relative velocity of point B with respect to the point A. Thus we confirm previous equation

\[ \mathbf{v}_{31}^B = \mathbf{v}_{32}^B + \mathbf{v}_{21}^B \]

3.5.3 The acceleration of the point B is a product of the velocity derivation

\[ \mathbf{a}_B^1 = \frac{d}{dt} \mathbf{v}_B^1 = \mathbf{a}_1^A + \mathbf{\alpha}_{21} \times (\mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA}) + \mathbf{\omega}_{21} \times (\mathbf{i}_2 x_2^{BA} + \mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA} + \mathbf{j}_2 y_2^{BA}) \]

\[ + \mathbf{i}_2 a^{BA}_{x_2} + \mathbf{i}_2 a^{BA}_{x_2} + \mathbf{j}_2 a^{BA}_{y_2} + \mathbf{j}_2 a^{BA}_{y_2} \]

\[ \mathbf{a}_B^1 = \mathbf{a}_1^A + \mathbf{\alpha}_{21} \times (\mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA}) + \mathbf{\omega}_{21} \times (\mathbf{i}_2 x_2^{BA} + \mathbf{j}_2 y_2^{BA}) + \mathbf{i}_2 a^{BA}_{x_2} + \mathbf{j}_2 a^{BA}_{y_2} \]

Thus

\[ + \mathbf{\omega}_{21} \times (\mathbf{i}_2 v^{BA}_{x_2} + \mathbf{j}_2 v^{BA}_{y_2}) + \mathbf{i}_2 a^{BA}_{x_2} + \mathbf{j}_2 a^{BA}_{y_2} \]
Where the expression
\[ a_{21}^B = a_1^A + a_{21}^A \times (i_{2x}^A + j_{2y}^A) + \omega_{21} \times [\omega_{21} \times (i_{2x}^A + j_{2y}^A)] \]
represents the acceleration of the driving motion at the point B
\[ a_{C}^B = 2\omega_{21} \times (i_{2x}^A + j_{2y}^A) \]
represents the Coriolis acceleration due to the driving angular motion and linear velocity of the relative motion.
\[ a_{32}^B = i_{2x}a_{x}^A + j_{2y}a_{y}^A \]
represents the acceleration of the relative motion at the point B.

Finally, we can conclude that while expressing the final acceleration by means of combined motion (driving and relative motion) of bodies a component called Coriolis acceleration has to be introduced.

### 3.5.4 Coriolis acceleration

Coriolis acceleration expressed in vector form:
\[ a_{C}^B = 2\omega_{21} \times v_{32}^B \]
And in matrix form:
\[ a_{C}^B = 2\Omega_{21}C_{21}v_{2}^A = 2\Omega_{21}v_{1}^A \]

Coriolis acceleration expresses the change of direction of relative velocity due to rotational driving motion and in the same time the change magnitude of driving velocity due to relative motion at the reference point.

Analysing the equation expressing the Coriolis acceleration:
The Coriolis acceleration has non zero value \( a_{C} \neq 0 \) if:
1) \( \omega_{dr} \neq 0 \) the driving motion exists in the form of rotation, GPM, spherical motion or GSM
2) \( v_{rel} \neq 0 \) the relative motion between bodies exists
3) \( \omega_{dr} \perp v_{rel} \) the angle between angular and linear velocities is different from 0, and \( \pi \)

The direction and orientation of Coriolis acceleration is given by rotating the relative velocity in the direction of the driving motion by an angle \( \pi/2 \).
3.5.5  Finding the pole of motion by means of combined motion

Analyzing the system:
B2 – RM
B3 – GPM
B4 – RM

Thus the body B2 and B4 are rotating around the fixed point of rotation \( O_{21} \) and \( O_{41} \) respectively. Thus the two points \( O_{21} \) and \( O_{41} \) are the poles of rotation for B2 and B4 respectively.

Thus the pole of B3 can be found in the intersection of two normals to the trajectory. The normal \( n_{\Delta} \) is given by two points - \( O_{21} \) and A. The point B draws universal plane curve with unknown center of curvature.

Thus applying the principle of combined motion we can write symbolically for motion of B3:

\[
31 = 32 + 21
\]

where \( 32 \) describes the pole of relative motion of B3 with respect to B2

and \( 21 \) describes the pole of driving motion of B2 with respect to the frame (B1)

Therefore applying poles of motion on B3 we get the direction of the normal that can be recorded symbolically as:

\[
n_{31} = O_{32} + O_{21}
\]

Second symbolic equation recording the combined motion of B3 is \( 31 = 34 + 41 \)

Thus the second normal for B3 is: \( n_{31} = O_{34} + O_{41} \)

Conclusion:

The pole of final motion, relative motion and driving motion lies on the same line.
3.6 SPHERICAL MOTION OF A BODY

Definition of spherical motion:
The body is moving with a spherical motion if one point of the body remains stationary at any instant.

The points on the body have constant distance from the center O, thus their trajectories are spherical curves, curves lying on spheres with common center O.

One point on the body remains stationary during the motion at any instance thus the new position of the body is given by a single rotation around the axis that passes through the stationary point.

This axis is called the instantaneous axis of rotation and coincides with the vector of total angular velocity. Points on instantaneous axis of rotation have zero linear velocity at the instant.

Thus the cone with the base radius \( r \) that is rolling on the plane \( \pi \) shares with this plane one single line at the instant the instantaneous axis of rotation. This line represents the contact region between the cone surface and the plane over which the cone is rolling.

Observing the cone motion on the plane we can describe the motion as a rotation around the cone axis of symmetry \( z_2 \)-axis (the natural axis of rotation) with angular velocity \( \phi \) and the rotation with angular velocity \( \psi \) around the axis perpendicular to the plane that passes through the stationary point on the body \( (y_1 \text{-axis}) \).

Associating the second coordinate system with moving body we can identify three angles called Euler’s angles:

\[
\dot{\vartheta} \quad \text{the nutation angle - describing the deviation of the natural axis of a body (} z_2 \text{) from the (} z_1 \text{) axis of the fixed coordinate system}
\]

\[
\psi \quad \text{the precession angle describes the change of position of natural axis of a cone with respect to } CS1
\]

\[
\phi \quad \text{the angle of natural rotation describes the change of position of a point on the body with respect to } CS2
\]

The total angular velocity is a resultant of angular velocity of nutation, precession, and natural rotation. The direction of the angular velocity coincides with the instantaneous axis of rotation (IA).

Thus:

\[
\mathbf{\omega} = \dot{\vartheta} + \psi + \phi
\]
The total angular velocity in rectangular $CS1$ is recorded as $\mathbf{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$

Since the angular velocities coincide with particular axes of rotation we need to transform them into $CS1$ thus receiving the components of total angular velocity:

$$\omega_x = \dot{\phi} \sin \psi \sin \vartheta + \dot{\psi} \cos \psi$$

$$\omega_y = -\dot{\phi} \sin \psi \cos \vartheta + \dot{\psi} \sin \vartheta$$

$$\omega_z = \dot{\phi} \cos \vartheta + \dot{\psi}$$

The angular acceleration: $ \mathbf{a} = \frac{d}{dt} \mathbf{\omega} = \frac{d}{dt} (\mathbf{e}_\omega \cdot \mathbf{\omega}) = \frac{d\mathbf{e}_\omega}{dt} \mathbf{\omega} + \mathbf{e}_\omega \frac{d\mathbf{\omega}}{dt}$

where $\frac{d\mathbf{e}_\omega}{dt} = \mathbf{\psi} \times \mathbf{\omega} = \mathbf{a}_1$ represents the change of the direction of the angular velocity $\mathbf{\omega}$

The direction of $\mathbf{\alpha}_1$ is perpendicular to the plane containing $\mathbf{\psi}$ and $\mathbf{e}_\omega$ while the orientation is given by the right hand rule.

The second component of angular acceleration $\mathbf{a}_2 = \mathbf{e}_\omega \frac{d\mathbf{\omega}}{dt}$ lies on the natural axis of rotation

Thus $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ therefore the direction of the final angular acceleration does not coincide with the direction of total angular velocity.

The total angular acceleration in rectangular $CS1$ is recorded as $\mathbf{a} = \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k}$
3.7 UNIVERSAL SPACE MOTION OF A BODY

Definition:

The trajectories of points on the body moving with universal space motion are a universal space curves. Thus the type of motion is referred to as universal space motion.

Similarly as we described universal planar motion, we can imagine that the body's final motion consists of body's translation and spherical motion, while the translation and rotation are the motions described with respect to reference point on the body.

If the point A is the stationary point during the spherical motion then we can select this point to be the suitable reference point while describing the final motion of the body.

Then the position of point M on the moving body is given as:

\[ \mathbf{r}_M = \mathbf{r}_i^A + \mathbf{r}_{MA} \]

and the velocity and acceleration in CS1 given in vector form:

\[ \mathbf{v}_M = \mathbf{v}_i^A + \omega \times \mathbf{r}_{MA} \]

\[ \mathbf{a}_M = \mathbf{a}_i^A + \alpha \times \mathbf{r}_{MA} + \omega \times (\omega \times \mathbf{r}_{MA}) \]

where \( \alpha \) and \( \omega \) are instantaneous kinematics quantities.

In matrix form:

\[ \mathbf{r}_M = \mathbf{r}_i^A + \mathbf{r}_{MA} = \mathbf{r}_i^A + \dot{\mathbf{C}}_{21} \mathbf{r}_{MA} \]

\[ \mathbf{v}_M = \mathbf{v}_i^A + \Omega_i \mathbf{C}_{21} \mathbf{r}_{MA} = \mathbf{v}_i^A + \Omega_i \mathbf{r}_{MA} \]

\[ \mathbf{a}_M = \mathbf{a}_i^A + \dot{\Omega}_i \mathbf{C}_{21} \mathbf{r}_{MA} + \Omega_i \dot{\mathbf{C}}_{21} \mathbf{r}_{MA} = \mathbf{a}_i^A + \dot{\mathbf{A}}_i \mathbf{r}_i^A + \Omega_i \mathbf{r}_{MA} \]
4 SYSTEM OF BODIES

The strategy of evaluating kinematical quantities for a single body can be extended as well for a system of bodies. As it was already mentioned the universal motion of a particular body can be by described by means of combined motion based on the relative and driving motion. Both motions, the driving and relative motion, could be of any type - translation, rotation, and universal planar motion, etc.

Thus the kinematical quantities for a system of bodies can be expressed in the same way. Prior to the description of the motion for a particular body it is useful if not necessary to analyse the whole system, describe the kinematical pair between bodies, identify the mobility of the system, identify the actuator of the system thus define the independent coordinate, and finally define the type of motion of each body in the system.

4.1 SIMULTANEOUS ROTATIONS AROUND CONCURRENT AXES

The body B3 rotates around its natural axis of rotation \( o_32 \) (the loci of all points that remain stationary with respect to B3), while the axis \( o_21 \) is positioned on the body B2 that rotates around its axis of rotation \( o_21 \).

Point O is stationary at any time therefore the body B3 moves with spherical motion that could be interpreted by symbolic equation as combined motion

\[
31 = 32 + 21
\]

Thus we can write for velocity of point M

\[
v_{31} = v_{32} + v_{21}
\]

represented in vector form by equation:

\[
v_{31} = \omega_{32} \times r_{31}^M + \omega_{21} \times r_{21}^M
\]

where

\[
v_{31} = \omega_{31} \times r_{31}^M
\]

and finally

\[
\omega_{31} = \omega_{32} + \omega_{21}
\]

Thus the final angular velocity is the sum of angular velocity of relative and driving motion. The vector of final angular velocity coincides with the instantaneous axis of rotation.

The total angular acceleration

\[
a_{31} = \frac{d}{dt} (\omega_{31}) = \frac{d}{dt} (e_{31} \cdot \omega_{31})
\]

Substituting for final angular velocity

\[
a_{31} = \frac{d}{dt} (\omega_{32} + \omega_{21}) = \frac{d}{dt} \omega_{32} + \frac{d}{dt} \omega_{21}
\]

Then the total angular acceleration is

\[
a_{31} = \frac{d}{dt} e_{32} \cdot \omega_{32} + \frac{d}{dt} e_{21} \cdot \omega_{21}
\]
Where \[
\frac{d}{dt}(e_{32} \cdot \omega_{32}) = \frac{de_{32}}{dt} \omega_{32} + e_{32} \frac{d\omega_{32}}{dt} = (\omega_{21} \times e_{32}) \omega_{32} + e_{32} \alpha_{32} = \omega_{21} \times \omega_{32} + \alpha_{32}
\]
and \[
\frac{d}{dt}(e_{21} \cdot \omega_{21}) = \mathbf{e}_{21} \frac{d\omega_{21}}{dt} = \mathbf{e}_{21} \cdot \alpha_{21} = \mathbf{a}_{21}
\]
therefore the final acceleration is given as
\[
\mathbf{a}_{31} = \mathbf{a}_{32} + \mathbf{a}_{21} + \omega_{21} \times \omega_{32}
\]
The last component in the equation is known as Resa le angular acceleration
\[
\mathbf{a}_{Res} = \omega_{21} \times \omega_{32}
\]
The condition for existence of Resal acceleration:
\[
\omega_{dr} \neq \mathbf{0}\\
\omega_{rel} \neq \mathbf{0}\\
\omega_{dr} \parallel \omega_{rel}
\]
The simultaneous rotations in matrix form:

**Angular velocity:**
\[
\mathbf{\omega}_{31} = \mathbf{\omega}_{32} + \mathbf{\omega}_{21} = \\
\begin{bmatrix}
\omega_{32x} \\
\omega_{32y} \\
\omega_{32z}
\end{bmatrix}
+ \\
\begin{bmatrix}
\omega_{21x} \\
\omega_{21y} \\
\omega_{21z}
\end{bmatrix}
\]

**Angular acceleration:**
\[
\mathbf{a}_{31} = \mathbf{a}_{32} + \mathbf{a}_{21} + \mathbf{a}_{res} = \\
\begin{bmatrix}
\alpha_{32x} \\
\alpha_{32y} \\
\alpha_{32z}
\end{bmatrix}
+ \\
\begin{bmatrix}
\alpha_{21x} \\
\alpha_{21y} \\
\alpha_{21z}
\end{bmatrix}
+ \mathbf{\Omega}_{21} \cdot \mathbf{\omega}_{32}
\]

### 4.2 Simultaneous Rotations Around Parallel Axes

Investigating kinematical quantities of a mechanism or part of it requires taking into consideration the whole set-up of bodies and constrains. Thus the first step is to analyse the mobility based on constrains, identify the actuator that controls the motion of the system, and identify the motion of each body.

Thus given system consist of:

B1 – fixed frame
B2 – rotating link
B3 – universal planar motion

The constrains limiting the motion of the system are two pins both attached to the B2

B2 rotates around its stationary centre of rotation O₁ and its points draw planar curves – concentric circles.

B3 rotates around the centre of rotation O₂ that connects B3 with B2
Thus the relative motion is the rotation of \( B_3 \) with respect to \( B_2 \) with angular velocity \( \omega_{32} \) and the driving motion is the rotation of \( B_2 \) with respect to the foundation \( B_1 \) with angular velocity \( \omega_{21} \).

Therefore:

\[
\phi_{31} \cdot \mathbf{k} = \phi_{32} \cdot \mathbf{k} + \phi_{21} \cdot \mathbf{k}
\]

with total angular velocity

\[
\alpha_{31} \cdot \mathbf{k} = \alpha_{32} \cdot \mathbf{k} + \alpha_{21} \cdot \mathbf{k}
\]

and angular acceleration

\[
\alpha_{31} \cdot \mathbf{k} = \alpha_{32} \cdot \mathbf{k} + \alpha_{21} \cdot \mathbf{k}
\]

since \( \alpha_{\text{Res}} = \omega_{21} \times \omega_{32} = 0 \)

Thus the kinematics quantities for a particular point \( B \) can be expressed in vector form:

- for position:
  \[
r_1^B = r_1^A + r_{1BA}
\]

- for velocity
  \[
  v_{31} = v_{32} + v_{21}
  \]
  \[
  v_{31} = \omega_{32} \times r_{1BA} + \omega_{21} \times r_1^B
  \]

- for acceleration:
  \[
  a_{31} = a_{32}^B + a_{21}^B + a_{\text{cor}}^B
  \]
  \[
  a_{31} = \left[ \omega_{32} \times \left( \omega_{32} \times r_{1BA}^A \right) + \alpha_{32} \times r_{1BA}^A \right] + \left[ \omega_{21} \times \left( \omega_{21} \times r_1^B \right) + \alpha_{21} \times r_1^B \right] + 2 \left( \omega_{21} \times v_{32}^A \right)
  \]

or in matrix form

- for position:
  \[
  r_1^B = r_1^A + r_{1BA}
  \]
  \[
  r_1^B = C_{21} \cdot r_2^A + C_{21} \cdot r_2^A \quad \text{where} \quad r_{2BA}^A = C_{32} \cdot r_3^A
  \]

- for velocity:
  \[
  v_{31}^B = \Omega_{21} \left( C_{21} \cdot r_2^A + C_{32} \cdot r_3^A \right) + C_{21} \cdot \Omega_{32} \cdot C_{32} \cdot r_3^A
  \]

- for acceleration:
  \[
  a_{31}^B = \left( C_{21} \cdot A_{32} + C_{32} \cdot \Omega_{32}^2 \right) \cdot r_{2}^A + \left( A_{21} + \Omega_{21}^2 \right) \cdot \left( r_1^A + r_{1BA}^A \right) + 2 \cdot \Omega_{21} \cdot C_{21} \cdot \Omega_{32} \cdot C_{32} \cdot r_3^A
  \]

Analysing the possibility of motion shows two cases:

1. Simultaneous rotations with the same orientation of angular velocity
2. Simultaneous rotation with angular velocities in opposite direction

In both cases the resulting motion is rotation with angular velocity \( \omega_{31} \)

Mechanical engineer faces problems related to simultaneous rotations around parallel axis in number of applications such as the gearbox, planetary gearbox, etc.